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Relativistic spherical functions on the Lorentz group

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Abstract

Matrix elements of irreducible representations of the Lorentz group are calculated on the basis of complex angular momentum. It is shown that Laplace–Beltrami operators, defined on this basis, give rise to Fuchsian differential equations. An explicit form of the matrix elements of the Lorentz group has been found via the addition theorem for generalized spherical functions. Different expressions of the matrix elements are given in terms of hypergeometric functions both for finite-dimensional and unitary representations of the principal and supplementary series of the Lorentz group.

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1. Introduction

As is known, an expansion problem of relativistic amplitudes requires the most simple form for the matrix elements of irreducible representations of the Lorentz group. Matrix elements of this group are studied for a long time by many authors. So, in 1956, Dolginov [1] (see also [2–4]) considered an analytic continuation of the Fock four-dimensional spherical functions (four-dimensional spherical functions of an Euclidean space were introduced by Fock [5] for the solution of the hydrogen atom problem in momentum representation). Basis functions, called *relativistic spherical functions* in the works [1–3], depend on angles of the radius vector in the four-dimensional spacetime. It should be noted that Dolginov–Toptygin relativistic spherical functions present the most degenerate form of the matrix elements of the Lorentz group. Different realizations of these elements were studied in the works [6–13]. The most complete form of the matrix elements of the Lorentz group was given in the works [8, 9] within the Gel'fand–Naimark basis [14, 15]. However, matrix elements in the Ström form, and also in the Sciarrino–Toller form [11], are very complicated and cumbersome. Smorodinsky and Huszar [16–18] found more simple and direct method for the definition of the matrix elements of the Lorentz group by means of a complexification of the three-dimensional rotation group and solution of the equation on eigenvalues of the Casimir operators (see also [19]).

In the present work matrix elements of irreducible representations of the Lorentz group are found on the basis of complex angular momentum ($SU(2) \otimes SU(2)$-basis). It is shown that

Laplace–Beltrami operators, defined in this basis, lead to Fuchsian differential equations which can be reduced to hypergeometric equations. An explicit form of the matrix elements has been found via the addition theorem for generalized spherical functions, where the functions P^l_{mn} and $\mathfrak{P}^l_{mn}$ are components. As is known [20], the matrix elements of $SU(2)$ are defined by the functions P^l_{mn}, and matrix elements of the group $QU(2)$ of quasiunitary matrices of the second order, which is isomorphic to the group $SL(2,\mathbb{R})$,[1] are expressed via the functions $\mathfrak{P}^l_{mn}$. The groups $SU(2)$ and $SU(1,1)$ are real forms of the group $SL(2,\mathbb{C})$. The factorization of the matrix elements of $SL(2,\mathbb{C})$ with respect to the subgroups $SU(2)$ and $SU(1,1)$ allows us to express these elements via the product of two hypergeometric functions both for finite-dimensional and unitary representations of the principal and supplementary series of the Lorentz group (it should be noted that matrix elements in the Ström form are expressed via the product of three hypergeometric functions). On the other hand, matrix elements of the Lorentz group play an essential role in quantum field theory on the Poincaré group [23–28], where the field operators are expressed via generalized Fourier integrals (it leads to harmonic analysis on the homogeneous spaces). Solutions of relativistic wave equations are reduced also to expansions in relativistic spherical functions [29, 30]. Moreover, the Biedenharn type relativistic wavefunctions [31] are defined completely in this framework [27, 28].

2. Relativistic spherical functions

As is known, the group $\mathrm{spin}_+(1,3) \simeq SL(2,\mathbb{C})$ is a universal covering of the proper orthochronous Lorentz group $SO_0(1,3)$. The group $SL(2,\mathbb{C})$ of all complex matrices

$$\mathfrak{g} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of second order with the determinant $\alpha\delta - \gamma\beta = 1$ is a *complexification* of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2,\mathbb{C})$. The transition from $SU(2)$ to $SL(2,\mathbb{C})$ is realized via the complexification of three real parameters φ, θ, ψ (Euler angles) of $SU(2)$. Let $\theta^c = \theta - i\tau$, $\varphi^c = \varphi - i\epsilon$, $\psi^c = \psi - i\varepsilon$ be complex Euler angles, where

$$\begin{aligned} &0 \leqslant \operatorname{Re}\theta^c = \theta \leqslant \pi, && -\infty < \operatorname{Im}\theta^c = \tau < +\infty, \\ &0 \leqslant \operatorname{Re}\varphi^c = \varphi < 2\pi, && -\infty < \operatorname{Im}\varphi^c = \epsilon < +\infty, \\ &-2\pi \leqslant \operatorname{Re}\psi^c = \psi < 2\pi, && -\infty < \operatorname{Im}\psi^c = \varepsilon < +\infty. \end{aligned} \tag{1}$$

Infinitesimal operators A_i and B_i of the group $SL(2,\mathbb{C})$ form a basis of the Lie algebra $sl(2,\mathbb{C})$ and satisfy the relations

$$\left.\begin{array}{lll} [\mathsf{A}_1,\mathsf{A}_2] = \mathsf{A}_3, & [\mathsf{A}_2,\mathsf{A}_3] = \mathsf{A}_1, & [\mathsf{A}_3,\mathsf{A}_1] = \mathsf{A}_2, \\ [\mathsf{B}_1,\mathsf{B}_2] = -\mathsf{A}_3, & [\mathsf{B}_2,\mathsf{B}_3] = -\mathsf{A}_1, & [\mathsf{B}_3,\mathsf{B}_1] = -\mathsf{A}_2, \\ [\mathsf{A}_1,\mathsf{B}_1] = 0, & [\mathsf{A}_2,\mathsf{B}_2] = 0, & [\mathsf{A}_3,\mathsf{B}_3] = 0, \\ [\mathsf{A}_1,\mathsf{B}_2] = \mathsf{B}_3, & [\mathsf{A}_1,\mathsf{B}_3] = -\mathsf{B}_2, & \\ [\mathsf{A}_2,\mathsf{B}_3] = \mathsf{B}_1, & [\mathsf{A}_2,\mathsf{B}_1] = -\mathsf{B}_3, & \\ [\mathsf{A}_3,\mathsf{B}_1] = \mathsf{B}_2, & [\mathsf{A}_3,\mathsf{B}_2] = -\mathsf{B}_1. & \end{array}\right\} \tag{2}$$

Let us consider the operators

$$\mathsf{X}_l = \tfrac{1}{2}i(\mathsf{A}_l + i\mathsf{B}_l), \qquad \mathsf{Y}_l = \tfrac{1}{2}i(\mathsf{A}_l - i\mathsf{B}_l), \qquad (l = 1,2,3). \tag{3}$$

Using relations (2), we obtain

$$[\mathsf{X}_k,\mathsf{X}_l] = i\varepsilon_{klm}\mathsf{X}_m, \qquad [\mathsf{Y}_l,\mathsf{Y}_m] = i\varepsilon_{lmn}\mathsf{Y}_n, \qquad [\mathsf{X}_l,\mathsf{Y}_m] = 0. \tag{4}$$

[1] Other designation of this group is $SU(1,1)$ also known as the three-dimensional Lorentz group, representations of which were studied by Bargmann [22].

Further, introducing generators of the form

$$\left.\begin{array}{ll}\mathsf{X}_+ = \mathsf{X}_1 + \mathrm{i}\mathsf{X}_2, & \mathsf{X}_- = \mathsf{X}_1 - \mathrm{i}\mathsf{X}_2,\\ \mathsf{Y}_+ = \mathsf{Y}_1 + \mathrm{i}\mathsf{Y}_2, & \mathsf{Y}_- = \mathsf{Y}_1 - \mathrm{i}\mathsf{Y}_2,\end{array}\right\} \tag{5}$$

we see that in virtue of commutativity of relations (4) a space of an irreducible finite-dimensional representation of the group $SL(2,\mathbb{C})$ can be spanned on the totality of $(2l+1)(2\dot{l}+1)$ basis vectors $|l,m;\dot{l},\dot{m}\rangle$, where $l, m, \dot{l}, \dot{m}$ are integer or half-integer numbers, $-l \leqslant m \leqslant l$, $-\dot{l} \leqslant \dot{m} \leqslant \dot{l}$. Therefore,

$$\begin{aligned}
&\mathsf{X}_-|l,m;\dot{l},\dot{m}\rangle = \sqrt{(l+m)(l-m+1)}|l,m-1,\dot{l},\dot{m}\rangle(m>-l),\\
&\mathsf{X}_+|l,m;\dot{l},\dot{m}\rangle = \sqrt{(l-m)(l+m+1)}|l,m+1;\dot{l},\dot{m}\rangle(m<l),\\
&\mathsf{X}_3|l,m;\dot{l},\dot{m}\rangle = m|l,m;\dot{l},\dot{m}\rangle,\\
&\mathsf{Y}_-|l,m;\dot{l},\dot{m}\rangle = \sqrt{(\dot{l}+\dot{m})(\dot{l}-\dot{m}+1)}|l,m;\dot{l},\dot{m}-1\rangle(\dot{m}>-\dot{l}),\\
&\mathsf{Y}_+|l,m;\dot{l},\dot{m}\rangle = \sqrt{(\dot{l}-\dot{m})(\dot{l}+\dot{m}+1)}|l,m;\dot{l},\dot{m}+1\rangle(\dot{m}<\dot{l}),\\
&\mathsf{Y}_3|l,m;\dot{l},\dot{m}\rangle = \dot{m}|l,m;\dot{l},\dot{m}\rangle.
\end{aligned} \tag{6}$$

From relations (4), it follows that each of the sets of infinitesimal operators X and Y generates the group $SU(2)$ and these two groups commute with each other. Thus, from relations (4) and (6) it follows that the group $SL(2,\mathbb{C})$, in essence, is equivalent locally to the group $SU(2)\otimes SU(2)$. Basis (6) was first introduced by Van der Waerden in [32].

On the group $SL(2,\mathbb{C})$ there exist the following Laplace–Beltrami operators:

$$\begin{aligned}
\mathsf{X}^2 &= \mathsf{X}_1^2 + \mathsf{X}_2^2 + \mathsf{X}_3^2 = \tfrac{1}{4}(\mathsf{A}^2 - \mathsf{B}^2 + 2\mathrm{i}\mathsf{AB}),\\
\mathsf{Y}^2 &= \mathsf{Y}_1^2 + \mathsf{Y}_2^2 + \mathsf{Y}_3^2 = \tfrac{1}{4}(\widetilde{\mathsf{A}}^2 - \widetilde{\mathsf{B}}^2 - 2\mathrm{i}\widetilde{\mathsf{A}}\widetilde{\mathsf{B}}).
\end{aligned} \tag{7}$$

At this point, we see that operators (7) contain the well-known Casimir operators $\mathsf{A}^2 - \mathsf{B}^2$, AB of the Lorentz group. Using expressions (1), we obtain a Euler parametrization of the Laplace–Beltrami operators:

$$\begin{aligned}
\mathsf{X}^2 &= \frac{\partial^2}{\partial\theta^{c2}} + \cot\theta^c\frac{\partial}{\partial\theta^c} + \frac{1}{\sin^2\theta^c}\left[\frac{\partial^2}{\partial\varphi^{c2}} - 2\cos\theta^c\frac{\partial}{\partial\varphi^c}\frac{\partial}{\partial\psi^c} + \frac{\partial^2}{\partial\psi^{c2}}\right],\\
\mathsf{Y}^2 &= \frac{\partial^2}{\partial\dot{\theta}^{c2}} + \cot\dot{\theta}^c\frac{\partial}{\partial\dot{\theta}^c} + \frac{1}{\sin^2\dot{\theta}^c}\left[\frac{\partial^2}{\partial\dot{\varphi}^{c2}} - 2\cos\dot{\theta}^c\frac{\partial}{\partial\dot{\varphi}^c}\frac{\partial}{\partial\dot{\psi}^c} + \frac{\partial^2}{\partial\dot{\psi}^{c2}}\right].
\end{aligned} \tag{8}$$

Here $\dot{\theta}^c = \theta + \mathrm{i}\tau$, $\dot{\varphi}^c = \varphi + \mathrm{i}\epsilon$, $\dot{\psi}^c = \psi + \mathrm{i}\varepsilon$ are complex conjugate Euler angles.

Matrix elements $t^l_{mn}(\mathfrak{g}) = \mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c)$ of irreducible representations of the group $SL(2,\mathbb{C})$ are eigenfunctions of operators (8):

$$\begin{aligned}
&[\mathsf{X}^2 + l(l+1)]\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c) = 0,\\
&[\mathsf{Y}^2 + \dot{l}(\dot{l}+1)]\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\dot{\varphi}^c,\dot{\theta}^c,\dot{\psi}^c) = 0,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\mathfrak{M}^l_{mn}(\mathfrak{g}) &= \mathrm{e}^{-\mathrm{i}(m\varphi^c + n\psi^c)}Z^l_{mn}(\cos\theta^c),\\
\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\mathfrak{g}) &= \mathrm{e}^{\mathrm{i}(\dot{m}\dot{\varphi}^c + \dot{n}\dot{\psi}^c)}Z^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^c).
\end{aligned} \tag{10}$$

Here $\mathfrak{M}^l_{mn}(\mathfrak{g})$ are general matrix elements of the representations of $SO_0(1,3)$, and $Z^l_{mn}(\cos\theta^c)$ are *hyperspherical functions*. Substituting functions (10) into (9) and taking into account operators (8) and substitutions $z = \cos\theta^c$, $\overset{*}{z} = \cos\dot{\theta}^c$, we arrive at the following differential equations:

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2}+l(l+1)\right]Z^l_{mn}(z)=0, \tag{11}$$

$$\left[(1-\overset{*}{z}{}^2)\frac{d^2}{d\overset{*}{z}{}^2}-2\overset{*}{z}\frac{d}{d\overset{*}{z}}-\frac{\dot{m}^2+\dot{n}^2-2\dot{m}\dot{n}\overset{*}{z}}{1-\overset{*}{z}{}^2}+\dot{l}(\dot{l}+1)\right]Z^{\dot{l}}_{\dot{m}\dot{n}}(\overset{*}{z})=0. \tag{12}$$

The latter equations have three singular points $-1, +1, \infty$. Equations (11), (12) are Fuchsian equations. Indeed, denoting $w(z)=Z^l_{mn}(z)$, we write equation (11) in the form

$$\frac{d^2w(z)}{dz^2}-p(z)\frac{dw(z)}{dz}+q(z)w(z)=0, \tag{13}$$

where

$$p(z)=\frac{2z}{(1-z)(1+z)}, \qquad q(z)=\frac{l(l+1)(1-z^2)-m^2-n^2+2mnz}{(1-z)^2(1+z)^2}.$$

Let us find solutions of (11). Applying the substitution

$$t=\frac{1-z}{2}, \qquad w(z)=t^{\frac{|m-n|}{2}}(1-t)^{\frac{|m+n|}{2}}v(t),$$

we arrive at hypergeometric equation

$$t(1-t)\frac{d^2v}{dt^2}+[c-(a+b+1)t]\frac{dv}{dt}-abv(t)=0, \tag{14}$$

where

$$\begin{aligned}
a&=l+1+\tfrac{1}{2}(|m-n|+|m+n|),\\
b&=-l+\tfrac{1}{2}(|m-n|+|m+n|),\\
c&=|m-n|+1.
\end{aligned}$$

Therefore, a solution of (14) is

$$v(t)=C_1{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right|t\right)+C_2t^{1-c}{}_2F_1\left(\left.\begin{matrix}b-c+1,a-c+1\\2-c\end{matrix}\right|t\right).$$

Coming back to initial variable, we obtain

$$\begin{aligned}
w(z)={}&C_1\left(\frac{1-z}{2}\right)^{\frac{|m-n|}{2}}\left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}}\\
&\times{}_2F_1\left(\left.\begin{matrix}l+1+\frac{1}{2}(|m-n|+|m+n|),-l+\frac{1}{2}(|m-n|+|m+n|)\\|m-n|+1\end{matrix}\right|\frac{1-z}{2}\right)\\
&+C_2\left(\frac{1-z}{2}\right)^{-\frac{|m-n|}{2}}\left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}}\\
&\times{}_2F_1\left(\left.\begin{matrix}-l+\frac{1}{2}(|m+n|-|m-n|),l+1+\frac{1}{2}(|m+n|-|m-n|)\\1-|m-n|\end{matrix}\right|\frac{1-z}{2}\right).
\end{aligned} \tag{15}$$

Carrying out the analogous calculations for equation (12), we find that

$$\begin{aligned}
w(\overset{*}{z})={}&C_1\left(\frac{1-\overset{*}{z}}{2}\right)^{\frac{|\dot{m}-\dot{n}|}{2}}\left(\frac{1+\overset{*}{z}}{2}\right)^{\frac{|\dot{m}+\dot{n}|}{2}}\\
&\times{}_2F_1\left(\left.\begin{matrix}l+1+\frac{1}{2}(|\dot{m}-\dot{n}|+|\dot{m}+\dot{n}|),-l+\frac{1}{2}(|\dot{m}-\dot{n}|+|\dot{m}+\dot{n}|)\\|\dot{m}-\dot{n}|+1\end{matrix}\right|\frac{1-\overset{*}{z}}{2}\right)
\end{aligned}$$

$$+ C_2 \left(\frac{1-\overset{*}{z}}{2}\right)^{-\frac{|\dot{m}-\dot{n}|}{2}} \left(\frac{1+\overset{*}{z}}{2}\right)^{\frac{|\dot{m}+\dot{n}|}{2}}$$

$$\times\, {}_2F_1\left(\begin{matrix} -l + \frac{1}{2}(|\dot{m}+\dot{n}| - |\dot{m}-\dot{n}|),\, l+1+\frac{1}{2}(|\dot{m}+\dot{n}| - |\dot{m}-\dot{n}|) \\ 1 - |\dot{m}-\dot{n}| \end{matrix} \,\middle|\, \frac{1-\overset{*}{z}}{2}\right). \tag{16}$$

As follows from (15) and (16), the functions Z^l_{mn} and $Z^{\dot{l}}_{\dot{m}\dot{n}}$ are expressed via the hypergeometric function. In virtue of the fast development of the theory of hypergeometric functions, the representations (15) and (16) are the most useful. Indeed, from (15) it follows that the function Z^l_{mn} can be represented by the following particular solution:

$$Z^l_{mn}(\cos\theta^c) = C_1 \sin^{|m-n|}\frac{\theta^c}{2}\cos^{|m+n|}\frac{\theta^c}{2}$$
$$\times\, {}_2F_1\left(\begin{matrix} l+1+\frac{1}{2}(|m-n|+|m+n|),\, -l+\frac{1}{2}(|m-n|+|m+n|) \\ |m-n|+1 \end{matrix}\,\middle|\, \sin^2\frac{\theta^c}{2}\right). \tag{17}$$

Let us now give a general definition for spherical functions on the group G. Let $T(g)$ be an irreducible representation of the group G in the space L and let H be a subgroup of G. The vector $\boldsymbol{\xi}$ in the space L is called *an invariant with respect to the subgroup* H if for all $h \in H$ the equality $T(h)\boldsymbol{\xi} = \boldsymbol{\xi}$ holds. The representation $T(g)$ is called *a representation of the class one with respect to the subgroup* H if in its space there are non-null vectors which are invariant with respect to H. At this point, a contraction of $T(g)$ onto its subgroup H is unitary:

$$(T(h)\boldsymbol{\xi}_1, T(h)\boldsymbol{\xi}_2) = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2).$$

Hence it follows that a function

$$f(g) = (T(g)\boldsymbol{\eta}, \boldsymbol{\xi})$$

corresponds the each vector $\boldsymbol{\eta} \in L$. $f(g)$ are called *spherical functions of the representation* $T(g)$ *with respect to* H.

Spherical functions can be considered as functions on homogeneous spaces $\mathcal{M} = G/H$. In its turn, a homogeneous space $\mathcal{M}$ of the group G has the following properties.

(a) It is a topological space on which the group G acts continuously, that is, let y be a point in $\mathcal{M}$, then gy is defined and is again a point in $\mathcal{M}$ ($g \in G$).
(b) This action is transitive, that is, for any two points y_1 and y_2 in $\mathcal{M}$ it is always possible to find a group element $g \in G$ such that $y_2 = gy_1$.

There is a one-to-one correspondence between the homogeneous spaces of G and the coset spaces of G. Let H_0 be a maximal subgroup of G which leaves the point y_0 invariant, $hy_0 = y_0, h \in H_0$, then H_0 is called the stabilizer of y_0. Representing now any group element of G in the form $g = g_c h$, where $h \in H_0$ and $g_c \in G/H_0$, we see that, by virtue of the transitivity property, any point $y \in \mathcal{M}$ can be given by $y = g_c h y_0 = g_c y$. Hence it follows that the elements g_c of the coset space give a parametrization of $\mathcal{M}$. The mapping $\mathcal{M} \leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on $\mathcal{M}$ is continuous by definition. The stabilizers H and H_0 of two different points y and y_0 are conjugate, since from $H_0 g_0 = g_0$, $y_0 = g^{-1}y$, it follows that $gH_0g^{-1}y = y$, that is, $H = gH_0g^{-1}$.

Coming back to the Lorentz group $G = SO_0(1,3)$, we see that there are the following homogeneous spaces of $SO_0(1,3)$ depending on the stabilizer H. First of all, when $H = 0$ the homogeneous space $\mathcal{M}_6$ coincides with *a group manifold* $\mathfrak{L}_6$ of $SO_0(1,3)$. Therefore, $\mathfrak{L}_6$ is a maximal homogeneous space of the Lorentz group. Further, when $H = \Omega^c_\psi$, where

$\Omega^{\rm c}_\psi$ is a group of diagonal matrices $\begin{pmatrix} {\rm e}^{\frac{{\rm i}\psi^{\rm c}}{2}} & 0 \\ 0 & {\rm e}^{-\frac{{\rm i}\psi^{\rm c}}{2}} \end{pmatrix}$, the homogeneous space $\mathcal{M}_4$ coincides with a two-dimensional complex sphere $S^{\rm c}_2$, $\mathcal{M}_4 = S^{\rm c}_2 \sim SL(2,\mathbb{C})/\Omega^{\rm c}_\psi$. The sphere $S^{\rm c}_2$ can be constructed from the quantities $z_k = x_k + {\rm i}y_k$, $\overset{*}{z}_k = x_k - {\rm i}y_k$ $(k = 1, 2, 3)$ as follows:

$$S^{\rm c}_2 : z_1^2 + z_2^2 + z_3^2 = \mathbf{x}^2 - \mathbf{y}^2 + 2{\rm i}\mathbf{xy} = r^2. \tag{18}$$

The complex conjugate (dual) sphere $\dot{S}^{\rm c}_2$ is

$$\dot{S}^{\rm c}_2 : \overset{*}{z}_1{}^2 + \overset{*}{z}_2{}^2 + \overset{*}{z}_3{}^2 = \mathbf{x}^2 - \mathbf{y}^2 - 2{\rm i}\mathbf{xy} = \overset{*}{r}{}^2. \tag{19}$$

The following homogeneous space $\mathcal{M}_3$ can be obtained when the stabilizer H coincides with a maximal compact subgroup $K = SO(3)$ of $SO_0(1,3)$. In this case we have a three-dimensional two-sheeted hyperboloid $\mathcal{M}_3 = H_3 \sim SO_0(1,3)/SO(3) \simeq SL(2,\mathbb{C})/SU(2)$, defined by the equation

$$H_3 = \{x \in \mathbb{R}^{1,3} \mid [x, x] = 1\}.$$

In the case $[x, x] = 0$ we arrive at a cone C_3 which can be considered also as a homogeneous space of $SO_0(1,3)$. Usually, only the upper sheets H_3^+ and C_3^+ are considered in applications.

Finally, a minimal homogeneous space $\mathcal{M}_2$ of $SO_0(1,3)$ is a two-dimensional real sphere $S_2 \sim SO(3)/SO(2)$. In contrast to the previous homogeneous spaces, the sphere S_2 coincides with a quotient space $SO_0(1,3)/P$, where P is a minimal parabolic subgroup of $SO_0(1,3)$. From the Iwasawa decompositions $SO_0(1,3) = KNA$ and $P = MNA$, where $M = SO(2)$, N and A are nilpotent and commutative subgroups of $SO_0(1,3)$, it follows that $SO_0(1,3)/P = KNA/MNA \sim K/M \sim SO(3)/SO(2)$.

Taking into account the list of homogeneous spaces of $SO_0(1,3)$, we introduce now the following types of spherical functions $f(\mathfrak{g})$ on the Lorentz group.

- $f(\mathfrak{g}) = \mathfrak{M}^l_{mn}(\mathfrak{g})$. This function is defined on the group manifold $\mathfrak{L}_6$ of $SO_0(1,3)$. It is the most general spherical function on the group $SO_0(1,3)$. In this case $f(\mathfrak{g})$ depends on all the six parameters of $SO_0(1,3)$ and for that reason it should be called *a function on the Lorentz group*. An explicit form of $\mathfrak{M}^l_{mn}(\mathfrak{g})$ (respectively $\mathfrak{M}^{\dot{l}}_{\dot{m}\dot{n}}(\mathfrak{g})$) for finite-dimensional representations and of $\mathfrak{M}^{-\frac{1}{2}+{\rm i}\rho}_{mn}(\mathfrak{g})$ (respectively $\mathfrak{M}^{-\frac{1}{2}-{\rm i}\rho}_{\dot{m}\dot{n}}(\mathfrak{g})$) for infinite-dimensional representations of $SO_0(1,3)$ will be given in sections 3 and 4, respectively.
- $f(\varphi^{\rm c}, \theta^{\rm c}) = \mathfrak{M}^m_l(\varphi^{\rm c}, \theta^{\rm c}, 0)$. This function is defined on the homogeneous space $\mathcal{M}_4 = S^{\rm c}_2 \sim SO_0(1,3)\big/\Omega^{\rm c}_\psi$, that is, on the surface of the two-dimensional complex sphere $S^{\rm c}_2$. The function $\mathfrak{M}^m_l(\varphi^{\rm c}, \theta^{\rm c}, 0)$ is a relativistic analogue of the usual spherical function $Y^m_l(\varphi, \theta)$ defined on the surface of the real 2-sphere S_2. In its turn, the function $f(\dot{\varphi}^{\rm c}, \dot{\theta}^{\rm c}) = \mathfrak{M}^{\dot{m}}_{\dot{l}}(\dot{\varphi}^{\rm c}, \dot{\theta}^{\rm c}, 0)$ is defined on the surface of the dual sphere $\dot{S}^{\rm c}_2$. General solutions of relativistic wave equations have been found via an expansion in spherical functions $f(\varphi^{\rm c}, \theta^{\rm c})$ [30]. An explicit form of the functions $\mathfrak{M}^m_l(\varphi^{\rm c}, \theta^{\rm c}, 0)$ $\big(\mathfrak{M}^{\dot{m}}_{\dot{l}}(\dot{\varphi}^{\rm c}, \dot{\theta}^{\rm c}, 0)\big)$ and $\mathfrak{M}^m_{-\frac{1}{2}+{\rm i}\rho}(\varphi^{\rm c}, \theta^{\rm c}, 0)$ $\big(\mathfrak{M}^{\dot{m}}_{-\frac{1}{2}-{\rm i}\rho}(\dot{\varphi}^{\rm c}, \dot{\theta}^{\rm c}, 0)\big)$ will be given in sections 3 and 4.
- $f(\epsilon, \tau, \varepsilon) = {\rm e}^{-{\rm i}m\epsilon}\mathfrak{P}^l_{mn}(\cosh\tau)\,{\rm e}^{-{\rm i}n\varepsilon}$. This function is defined on the homogeneous space $\mathcal{M}_3 = H_3^+ \sim SO_0(1,3)/SO(3)$, that is, on the upper sheet of the hyperboloid $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1$. In essence, we come here to representations of $SO_0(1,3)$ restricted to the subgroup $SU(1,1)$ [11, 13].
- $f(\varphi, \theta, \psi) = {\rm e}^{-{\rm i}m\varphi}P^l_{mn}(\cos\theta)\,{\rm e}^{-{\rm i}n\psi}$. This function is defined on the homogeneous space $\mathcal{M}_2 = S_2 \sim SO(3)/SO(2)$, that is, on the surface of the two-dimensional real sphere S_2. We come here to the most degenerate representations of $SO_0(1,3)$ restricted to the subgroup $SU(2)$.

We see that only the first two functions $f(\mathfrak{g})$ and $f(\varphi^c, \theta^c)$ can be considered as functions on the Lorentz group $SO_0(1,3)$; other two functions $f(\epsilon, \tau, \varepsilon)$ and $f(\varphi, \theta, \psi)$ present degenerate cases corresponding to the subgroups $SU(1,1)$ and $SU(2)$. For that reason the functions $f(\mathfrak{g})$ and $f(\varphi^c, \theta^c)$ should be called *relativistic spherical functions on the Lorentz group*.

3. Hyperspherical functions and addition theorem for generalized spherical functions

In this section, we will find expressions for the matrix elements (relativistic spherical functions) containing explicitly all six parameters of the Lorentz group. Moreover, such a form of the matrix elements are the most suitable for forthcoming tasks of harmonic analysis on the Lorentz and Poincaré groups.

As is known, the groups $SU(2)$ and $SU(1,1) \simeq SL(2,\mathbb{R})$ are real forms of $SL(2,\mathbb{C})$. As a direct consequence of this, a structure of the matrix elements of these groups is very similar to a corresponding structure of matrix elements for the group $SL(2,\mathbb{C})$. Indeed, matrix elements of irreducible representations of $SU(2)$ have the form [21, 33]

$$t^l_{mn}(u) = \mathrm{e}^{-\mathrm{i}m\varphi} P^l_{mn}(\cos\theta)\,\mathrm{e}^{-\mathrm{i}n\psi},$$

where

$$P^l_{mn}(\cos\theta) = \mathrm{i}^{m-n}\sqrt{\frac{\Gamma(l-m+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l+n+1)}}\cos^{m+n}\frac{\theta}{2}\sin^{m-n}\frac{\theta}{2}$$
$$\times\sum_{t=0}^{l-m}\frac{(-1)^t\Gamma(l+m+t+1)}{\Gamma(t+1)\Gamma(m-n+t+1)\Gamma(l-m-t+1)}\sin^{2t}\frac{\theta}{2}. \tag{20}$$

Here φ, θ, ψ are real Euler parameters for $SU(2)$ (the first column from relations (1)). At $m \geqslant n$ the function $P^l_{mn}(\cos\theta)$ is expressed via the hypergeometric function as

$$P^l_{mn}(\cos\theta) = \frac{\mathrm{i}^{m-n}}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(l-n+1)\Gamma(l+m+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}$$
$$\times\cos^{m+n}\frac{\theta}{2}\sin^{m-n}\frac{\theta}{2}\,{}_2F_1\left(\left.\begin{matrix}l+m+1, m-l\\ m-n+1\end{matrix}\right|\sin^2\frac{\theta}{2}\right). \tag{21}$$

Analogously, at $n \geqslant m$

$$P^l_{mn}(\cos\theta) = \frac{\mathrm{i}^{n-m}}{\Gamma(n-m+1)}\sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}}$$
$$\times\cos^{m+n}\frac{\theta}{2}\sin^{n-m}\frac{\theta}{2}\,{}_2F_1\left(\left.\begin{matrix}l+n+1, n-l\\ n-m+1\end{matrix}\right|\sin^2\frac{\theta}{2}\right). \tag{22}$$

It is easy to see that functions (21) and (22) with an accuracy of the constant coincide with function (15) (correspondingly (17)) if we open the modules and suppose $z = \cos\theta$.

Other expression for the function $P^l_{mn}(\cos\theta)$, related to (20), is defined by the transformation $u = kz$, where $k = \begin{pmatrix}\bar{\alpha}^{-1} & \beta\\ 0 & \bar{\alpha}\end{pmatrix}$ and $z = \begin{pmatrix}1 & 0\\ -\bar{\beta}/\bar{\alpha} & 1\end{pmatrix}$. This expression has the form

$$P^l_{mn}(\cos\theta) = \mathrm{i}^{m-n}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}\cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}$$
$$\times\sum_{j=\max(0,n-m)}^{\min(l-m,l+n)}\frac{\mathrm{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+n-j+1)\Gamma(m-n+j+1)}. \tag{23}$$

Correspondingly, functions (23) are expressed via the hypergeometric function as follows:

$$P^l_{mn}(\cos\theta) = \mathrm{i}^{m-n}\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}$$
$$\times\cos^{2l}\frac{\theta}{2}\tan^{m-n}\frac{\theta}{2}{}_2F_1\left(\begin{matrix}m-l,-n-l\\ m-n+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right),\qquad m\geqslant n; \tag{24}$$

$$P^l_{mn}(\cos\theta) = \mathrm{i}^{n-m}\sqrt{\frac{\Gamma(l+n+1)\Gamma(l-m+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}}$$
$$\times\cos^{2l}\frac{\theta}{2}\tan^{n-m}\frac{\theta}{2}{}_2F_1\left(\begin{matrix}n-l,-m-l\\ n-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right),\qquad n\geqslant m. \tag{25}$$

In turn, matrix elements of irreducible representations of the group $SU(1,1)$ have the form [21, 33]

$$t^l_{mn}(g) = \mathrm{e}^{-\mathrm{i}m\epsilon}\mathfrak{P}^l_{mn}(\cosh\tau)\,\mathrm{e}^{-\mathrm{i}n\varepsilon},$$

where in the case of finite-dimensional representations

$$\mathfrak{P}^l_{mn}(\cosh\tau) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l-n+1)}{\Gamma(l+m+1)\Gamma(l+n+1)}}\cosh^{m+n}\frac{\theta}{2}\sinh^{m+n}\frac{\theta}{2}$$
$$\times\sum_{s=0}^{l-m}\frac{(-1)^s\Gamma(l+m+s+1)}{\Gamma(s+1)\Gamma(m-n+s+1)\Gamma(l-m-s+1)}\sinh^{2s}\frac{\theta}{2}, \tag{26}$$

or

$$\mathfrak{P}^l_{mn}(\cosh\tau) = \sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-n+1)\Gamma(l+n+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{m-n}\frac{\tau}{2}$$
$$\times\sum_{s=\max(0,n-m)}^{\min(l-m,l+n)}\frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-m-s+1)\Gamma(l+n-s+1)\Gamma(m-n+s+1)}. \tag{27}$$

Here ϵ, τ, ε are real Euler parameters for the group $SU(1,1)$ (the second column from (1) at the restriction of the parameters ϵ and ε within the limits $0\leqslant\epsilon\leqslant 2\pi$ and $-2\pi\leqslant\varepsilon<2\pi$). The functions $\mathfrak{P}^l_{mn}(\cosh\tau)$ can be reduced also to hypergeometric functions. So, at $m\geqslant n$ we have

$$\mathfrak{P}^l_{mn}(\cosh\tau) = \frac{1}{\Gamma(m-n+1)}\sqrt{\frac{\Gamma(l-n+1)\Gamma(l+m+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\cosh^{m+n}\frac{\tau}{2}\sinh^{m-n}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}l+m+1,m-l\\ m-n+1\end{matrix}\middle|-\sinh^2\frac{\tau}{2}\right) = \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}$$
$$\times\cosh^{2l}\frac{\tau}{2}\tanh^{m-n}\frac{\tau}{2}{}_2F_1\left(\begin{matrix}m-l,-n-l\\ m-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right). \tag{28}$$

Correspondingly, at $n\geqslant m$

$$\mathfrak{P}^l_{mn}(\cosh\tau) = \frac{1}{\Gamma(n-m+1)}\sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l-n+1)\Gamma(l+m+1)}}\cosh^{m+n}\frac{\tau}{2}\sinh^{n-m}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}l+n+1,n-l\\ n-m+1\end{matrix}\middle|-\sinh^2\frac{\tau}{2}\right) = \sqrt{\frac{\Gamma(l+n+1)\Gamma(l-m+1)}{\Gamma(l-m+1)\Gamma(l+m+1)}}$$
$$\times\cosh^{2l}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}{}_2F_1\left(\begin{matrix}n-l,-m-l\\ n-m+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right). \tag{29}$$

In the case of principal series of unitary representations, matrix elements are (see, for example, [21, 34])

$$\begin{aligned}\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(\cosh\tau) &= \sqrt{\Gamma\left(i\rho-n+\tfrac{1}{2}\right)\Gamma\left(i\rho+n+\tfrac{1}{2}\right)\Gamma\left(i\rho-m+\tfrac{1}{2}\right)\Gamma\left(i\rho+m+\tfrac{1}{2}\right)}\\ &\quad\times\cosh^{2i\rho-1}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\sum_{s=\max(0,m-n)}^{\infty}\\ &\quad\times\frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma\left(i\rho-n-s+\tfrac{1}{2}\right)\Gamma(n-m+s+1)\Gamma\left(i\rho+m-s+\tfrac{1}{2}\right)},\end{aligned}\tag{30}$$

or

$$\begin{aligned}\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(\cosh\tau) &= \sqrt{\frac{\Gamma\left(i\rho+m+\tfrac{1}{2}\right)\Gamma\left(i\rho-n+\tfrac{1}{2}\right)}{\Gamma\left(i\rho-m+\tfrac{1}{2}\right)\Gamma\left(i\rho+n+\tfrac{1}{2}\right)}}\\ &\quad\times\cosh^{2i\rho-1}\frac{\tau}{2}\tanh^{m-n}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix}m-i\rho+\frac{1}{2},-n-i\rho+\frac{1}{2}\\ m-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\end{aligned}\tag{31}$$

at $m \geqslant n$ and

$$\begin{aligned}\mathfrak{P}_{mn}^{-\frac{1}{2}+i\rho}(\cosh\tau) &= \sqrt{\frac{\Gamma\left(i\rho+n+\tfrac{1}{2}\right)\Gamma\left(i\rho-m+\tfrac{1}{2}\right)}{\Gamma\left(i\rho-n+\tfrac{1}{2}\right)\Gamma\left(i\rho+m+\tfrac{1}{2}\right)}}\\ &\quad\times\cosh^{2i\rho-1}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix}n-i\rho+\frac{1}{2},-m-i\rho+\frac{1}{2}\\ n-m+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\end{aligned}\tag{32}$$

at $n \geqslant m$.

As is known [21], generalized spherical functions $P^l_{mn}(\cos\theta)$ satisfy the following addition theorem:

$$e^{-i(m\varphi+n\psi)}P^l_{mn}(\cos\theta)=\sum_{k=-l}^{l}e^{-ik\varphi_2}P^l_{mk}(\cos\theta_1)P^l_{kn}(\cos\theta_2),\tag{33}$$

where the angles φ, ψ, θ, θ_1, φ_2, θ_2 are related by the formulae

$$\cos\theta=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\varphi_2,\tag{34}$$

$$e^{i\varphi}=\frac{\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\cos\varphi+2+\sin\theta_2\sin\varphi_2}{\sin\theta},\tag{35}$$

$$e^{\frac{i(\varphi+\psi)}{2}}=\frac{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\frac{\varphi_2}{2}}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{-i\frac{\varphi_2}{2}}}{\cos\frac{\theta}{2}}.\tag{36}$$

Let $\cos(\theta-i\tau)$ and $\varphi_2=0$, then formulae (34)–(36) take the form

$$\begin{aligned}\cos\theta^c &= \cos\theta\cosh\tau+i\sin\theta\sinh\tau,\\ e^{i\varphi} &= \frac{\sin\theta\cosh\tau-i\cos\theta\sinh\tau}{\sin\theta^c}=1,\\ e^{\frac{i(\varphi+\psi)}{2}} &= \frac{\cos\frac{\theta}{2}\cosh\frac{\tau}{2}+i\sin\frac{\theta}{2}\sinh\frac{\tau}{2}}{\cos\frac{\theta^c}{2}}=1.\end{aligned}$$

Hence it follows that $\varphi=\psi=0$ and formula (33) can be written as

$$Z^l_{mn}(\cos\theta^c)=\sum_{k=-l}^{l}P^l_{mk}(\cos\theta)\mathfrak{P}^l_{kn}(\cosh\tau).$$

Therefore, using the addition theorem, we derived a new representation for the hyperspherical function. Further, taking into account (23) and (27), we obtain an explicit expression for $Z^l_{mn}(\cos\theta^c)$,

$$Z^l_{mn}(\cos\theta^c) = \sum_{k=-l}^{l} \mathrm{i}^{m-k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2}$$
$$\times \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{\mathrm{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}$$
$$\times \sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cosh^{2l}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}. \tag{37}$$

By way of example let us calculate matrix elements $\mathfrak{M}^l_{mn}(\mathfrak{g}) = \mathrm{e}^{-m\varphi^c} Z^l_{mn}(\cos\theta^c)\,\mathrm{e}^{-\mathrm{i}n\psi^c}$ at $l = 0, 1/2, 1$, where $Z^l_{mn}(\cos\theta^c)$ is defined via (37). The matrices of finite-dimensional representations at $l = 0, \frac{1}{2}, 1$ have the following form:

$$T_0(\varphi^c,\theta^c,\psi^c) = 1, \tag{38}$$

$$T_{\frac12}(\varphi^c,\theta^c,\psi^c) = \begin{pmatrix} \mathfrak{M}^{\frac12}_{-\frac12-\frac12} & \mathfrak{M}^{\frac12}_{\frac12-\frac12} \\ \mathfrak{M}^{\frac12}_{-\frac12\frac12} & \mathfrak{M}^{\frac12}_{\frac12\frac12} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi^c} Z^{\frac12}_{-\frac12-\frac12}\mathrm{e}^{\frac{\mathrm{i}}{2}\psi^c} & \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi^c} Z^{\frac12}_{-\frac12\frac12}\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi^c} \\ \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi^c} Z^{\frac12}_{\frac12-\frac12}\mathrm{e}^{\frac{\mathrm{i}}{2}\psi^c} & \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi^c} Z^{\frac12}_{\frac12\frac12}\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi^c} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi^c}\cos\frac{\theta^c}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\psi^c} & \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\varphi^c}\sin\frac{\theta^c}{2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi^c} \\ \mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi^c}\sin\frac{\theta^c}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\psi^c} & \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi^c}\cos\frac{\theta^c}{2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi^c} \end{pmatrix}$$
$$= \begin{pmatrix} \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2} + \mathrm{i}\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]\mathrm{e}^{\frac{\epsilon+\varepsilon+\mathrm{i}(\varphi+\psi)}{2}} & \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2} + \mathrm{i}\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]\mathrm{e}^{\frac{\epsilon-\varepsilon+\mathrm{i}(\varphi-\psi)}{2}} \\ \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2} + \mathrm{i}\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\right]\mathrm{e}^{\frac{\varepsilon-\epsilon+\mathrm{i}(\psi-\varphi)}{2}} & \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2} + \mathrm{i}\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\right]\mathrm{e}^{\frac{-\epsilon-\varepsilon-\mathrm{i}(\varphi+\psi)}{2}} \end{pmatrix}, \tag{39}$$

$$T_1(\varphi^c,\theta^c,\psi^c) = \begin{pmatrix} \mathfrak{M}^1_{-1-1} & \mathfrak{M}^1_{-10} & \mathfrak{M}^1_{-11} \\ \mathfrak{M}^1_{0-1} & \mathfrak{M}^1_{00} & \mathfrak{M}^1_{01} \\ \mathfrak{M}^1_{1-1} & \mathfrak{M}^1_{10} & \mathfrak{M}^1_{11} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi^c} Z^1_{-1-1}\mathrm{e}^{\mathrm{i}\psi^c} & \mathrm{e}^{\mathrm{i}\varphi^c} Z^1_{-10} & \mathrm{e}^{\mathrm{i}\varphi^c} Z^1_{-11}\mathrm{e}^{-\mathrm{i}\psi^c} \\ Z^1_{0-1}\mathrm{e}^{\mathrm{i}\psi^c} & Z^1_{00} & Z^1_{01}\mathrm{e}^{-\mathrm{i}\psi^c} \\ \mathrm{e}^{-\mathrm{i}\varphi^c} Z^1_{1-1}\mathrm{e}^{\mathrm{i}\psi^c} & \mathrm{e}^{-\mathrm{i}\psi^c} Z^1_{10} & \mathrm{e}^{-\mathrm{i}\varphi^c} Z^1_{11}\mathrm{e}^{-\mathrm{i}\psi^c} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi^c}\cos^2\frac{\theta^c}{2}\mathrm{e}^{\mathrm{i}\psi^c} & \frac{\mathrm{i}}{\sqrt2}\mathrm{e}^{\mathrm{i}\varphi^c}\sin\theta^c & -\mathrm{e}^{\mathrm{i}\varphi^c}\sin^2\frac{\theta^c}{2}\mathrm{e}^{-\mathrm{i}\psi^c} \\ \frac{\mathrm{i}}{\sqrt2}\sin\theta^c\mathrm{e}^{\mathrm{i}\psi^c} & \cos\theta^c & \frac{\mathrm{i}}{\sqrt2}\sin\theta^c\mathrm{e}^{-\mathrm{i}\psi^c} \\ -\mathrm{e}^{-\mathrm{i}\varphi^c}\sin^2\frac{\theta^c}{2}\mathrm{e}^{\mathrm{i}\psi^c} & \frac{\mathrm{i}}{\sqrt2}\mathrm{e}^{-\mathrm{i}\varphi^c}\sin\theta^c & \mathrm{e}^{-\mathrm{i}\varphi^c}\cos^2\frac{\theta^c}{2}\mathrm{e}^{-\mathrm{i}\psi^c} \end{pmatrix}$$
$$= \left(\begin{matrix} \left[\cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2} + \frac{\mathrm{i}\sin\theta\sinh\tau}{2} - \sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}\right]\mathrm{e}^{\epsilon+\varepsilon+\mathrm{i}(\varphi+\psi)} \\ \left[\frac{1}{\sqrt2}(\cos\theta\sinh\tau + \mathrm{i}\sin\theta\cosh\tau)\right]\mathrm{e}^{\varepsilon+\mathrm{i}\psi} \\ \left[\cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2} + \frac{\mathrm{i}\sin\theta\sinh\tau}{2} - \sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}\right]\mathrm{e}^{\varepsilon-\epsilon+\mathrm{i}(\psi-\varphi)} \end{matrix}\right.$$
$$\begin{matrix} \left[\frac{1}{\sqrt2}(\cos\theta\sinh\tau + \mathrm{i}\sin\theta\cosh\tau)\right]\mathrm{e}^{\epsilon+\mathrm{i}\varphi} \\ \cos\theta\cosh\tau + \mathrm{i}\sin\theta\sinh\tau \\ \left[\frac{1}{\sqrt2}(\cos\theta\sinh\tau + \mathrm{i}\sin\theta\cosh\tau)\right]\mathrm{e}^{-\epsilon-\mathrm{i}\varphi} \end{matrix}$$

$$\begin{array}{c}\left[\cos^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}+\frac{\mathrm{i}\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}\right]\mathrm{e}^{\epsilon-\varepsilon+\mathrm{i}(\varphi-\psi)}\\ \left[\frac{1}{\sqrt{2}}(\cos\theta\sinh\tau+\mathrm{i}\sin\theta\cosh\tau)\right]\mathrm{e}^{-\varepsilon-\mathrm{i}\psi}\\ \left[\cos^2\frac{\theta}{2}\cosh^2\frac{\tau}{2}+\frac{\mathrm{i}\sin\theta\sinh\tau}{2}-\sin^2\frac{\theta}{2}\sinh^2\frac{\tau}{2}\right]\mathrm{e}^{-\epsilon-\varepsilon-\mathrm{i}(\varphi+\psi)}\end{array}\Bigg). \tag{40}$$

Let us now express $Z^l_{mn}(\cos\theta^c)$ via the hypergeometric functions. So, at $m \geqslant n$ from (24) and (28) it follows that

$$Z^l_{mn}(\cos\theta^c)=\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\mathrm{i}^{m-k}\tan^{m-k}\frac{\theta}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}m-l,-k-l\\ m-k+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-n-l\\ k-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right). \tag{41}$$

Analogously, at $n \geqslant m$ from formulae (25) and (29) we have

$$Z^l_{mn}(\cos\theta^c)=\sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}}\cos^{2l}\frac{\theta}{2}\cosh^{2l}\frac{\tau}{2}\sum_{k=-l}^{l}\mathrm{i}^{k-m}\tan^{k-m}\frac{\theta}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}k-l,-m-l\\ k-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}n-l,-k-l\\ n-k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right). \tag{42}$$

Other representation for $Z^l_{mn}(\cos\theta^c)$ in the form of hypergeometric function can be obtained from formulae (21), (22) and (28), (29). Namely,

$$Z^l_{mn}(\cos\theta^c)=\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\sum_{k=-l}^{l}\frac{\mathrm{i}^{m-k}}{\Gamma(m-k+1)\Gamma(k-n+1)}$$
$$\times\cos^{m+k}\frac{\theta}{2}\sin^{m-k}\frac{\theta}{2}\sinh^{k-n}\frac{\tau}{2}\cosh^{k+n}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}l+m+1,m-l\\ m-k+1\end{matrix}\middle|\sin^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k+l+1,k-l\\ k-n+1\end{matrix}\middle|-\sinh^2\frac{\tau}{2}\right),\quad m\geqslant n;$$

$$Z^l_{mn}(\cos\theta^c)=\sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}}\sum_{k=-l}^{l}\frac{\mathrm{i}^{k-m}}{\Gamma(k-m+1)\Gamma(n-k+1)}$$
$$\times\cos^{m+k}\frac{\theta}{2}\sin^{k-m}\frac{\theta}{2}\sinh^{n-k}\frac{\tau}{2}\cosh^{k+n}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}l+k+1,k-l\\ k-m+1\end{matrix}\middle|\sin^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}n+l+1,n-l\\ n-k+1\end{matrix}\middle|-\sinh^2\frac{\tau}{2}\right),\quad n\geqslant m.$$

Analogous expressions take place for the functions $Z^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^c)$.

Relativistic spherical functions of the second type $f(\varphi^c,\theta^c)=\mathfrak{M}^m_l(\varphi^c,\theta^c,0)=\mathrm{e}^{-\mathrm{i}m\varphi^c}Z^m_l(\cos\theta^c)$, where

$$Z^m_l(\cos\theta^c)=\sum_{k=-l}^{l}P^l_{mk}(\cos\theta)\mathfrak{P}^k_l(\cosh\tau),$$

are defined on the surface of complex 2-sphere (18). In its turn, the functions $f(\dot{\varphi}^c,\dot{\theta}^c)=\mathrm{e}^{\mathrm{i}\dot{m}\dot{\varphi}^c}Z^{\dot{m}}_{\dot{l}}(\cos\dot{\theta}^c)$ are defined on the surface of dual sphere (19). Explicit expressions and hypergeometric type formulae for $f(\varphi^c,\theta^c)$ and $f(\dot{\varphi}^c,\dot{\theta}^c)$ follow directly from the previous expressions for $f(\mathfrak{g})$ at $n=0$.

4. Relativistic spherical functions of unitary representations of $SO_0(1,3)$

Relativistic spherical functions $\mathfrak{M}^l_{mn}(\varphi^c,\theta^c,\psi^c)$, considered in the previous sections, define matrix elements of non-unitary finite-dimensional representations of the group $SO_0(1,3)$. As is known [14], finite-dimensional (spinor) representations of $SO_0(1,3)$ in the space of symmetric polynomials $\mathrm{Sym}_{(k,r)}$ have the following form:

$$T_{\mathfrak{g}}q(\xi,\overline{\xi}) = (\gamma\xi+\delta)^{l_0+l_1-1}\overline{(\gamma\xi+\delta)}^{l_0-l_1+1}q\left(\frac{\alpha\xi+\beta}{\gamma\xi+\delta};\overline{\frac{\alpha\xi+\beta}{\gamma\xi+\delta}}\right), \tag{43}$$

where $k=l_0+l_1-1$, $r=l_0-l_1+1$, and the pair (l_0,l_1) defines some representation of $SO_0(1,3)$ in the Gel'fand–Naimark basis:

$$\begin{aligned}
&H_3\xi_{k\nu} = m\xi_{k\nu},\\
&H_+\xi_{k\nu} = \sqrt{(k+\nu+1)(k-\nu)}\,\xi_{k,\nu+1},\\
&H_-\xi_{k\nu} = \sqrt{(k+\nu)(k-\nu+1)}\,\xi_{k,\nu-1},\\
&F_3\xi_{k\nu} = C_l\sqrt{k^2-\nu^2}\,\xi_{k-1,\nu} - A_l\nu\xi_{k,\nu} - C_{k+1}\sqrt{(k+1)^2-\nu^2}\,\xi_{k+1,\nu},\\
&F_+\xi_{k\nu} = C_k\sqrt{(k-\nu)(k-\nu-1)}\,\xi_{k-1,\nu+1} - A_k\sqrt{(k-\nu)(k+\nu+1)}\,\xi_{k,\nu+1}\\
&\qquad\qquad + C_{k+1}\sqrt{(k+\nu+1)(k+\nu+2)}\,\xi_{k+1,\nu+1},\\
&F_-\xi_{k\nu} = -C_k\sqrt{(k+\nu)(k+\nu-1)}\,\xi_{k-1,\nu-1} - A_k\sqrt{(k+\nu)(k-\nu+1)}\,\xi_{k,\nu-1}\\
&\qquad\qquad - C_{k+1}\sqrt{(k-\nu+1)(k-\nu+2)}\,\xi_{k+1,\nu-1},
\end{aligned}$$

$$A_k = \frac{\mathrm{i}l_0l_1}{k(k+1)}, \qquad C_k = \frac{\mathrm{i}}{k}\sqrt{\frac{(k^2-l_0^2)(k^2-l_1^2)}{4k^2-1}}, \tag{44}$$

$$\nu = -k, -k+1, \ldots, k-1, k, \qquad k = l_0, l_0+1, \ldots,$$

where l_0 is positive integer or half-integer number and l_1 is an arbitrary complex number. These formulae define a finite-dimensional representation of the group $SO_0(1,3)$ when $l_1^2=(l_0+p)^2$, p is some natural number. In the case $l_1^2\neq(l_0+p)^2$ we have an infinite-dimensional representation of $SO_0(1,3)$. The operators $H_3, H_+, H_-, F_3, F_+, F_-$ are

$$\begin{aligned}
H_+ &= \mathrm{i}\mathsf{A}_1 - \mathsf{A}_2, & H_- &= \mathrm{i}\mathsf{A}_1 + \mathsf{A}_2, & H_3 &= \mathrm{i}\mathsf{A}_3,\\
F_+ &= \mathrm{i}\mathsf{B}_1 - \mathsf{B}_2, & F_- &= \mathrm{i}\mathsf{B}_1 + \mathsf{B}_2, & F_3 &= \mathrm{i}\mathsf{B}_3.
\end{aligned}$$

This basis was first given by Gel'fand in 1944 (see also [15, 35, 36]). The following relations between generators $\mathsf{Y}_\pm, \mathsf{X}_\pm, \mathsf{Y}_3, \mathsf{X}_3$ and $H_\pm, F_\pm, H_3, F_3$ define a relationship between the Van der Waerden and Gel'fand–Naimark bases:

$$\begin{aligned}
\mathsf{Y}_+ &= -\tfrac{1}{2}(F_+ + \mathrm{i}H_+), & \mathsf{X}_+ &= \tfrac{1}{2}(F_+ - \mathrm{i}H_+),\\
\mathsf{Y}_- &= -\tfrac{1}{2}(F_- + \mathrm{i}H_-), & \mathsf{X}_- &= \tfrac{1}{2}(F_- - \mathrm{i}H_-),\\
\mathsf{Y}_3 &= -\tfrac{1}{2}(F_3 + \mathrm{i}H_3), & \mathsf{X}_3 &= \tfrac{1}{2}(F_3 - \mathrm{i}H_3).
\end{aligned}$$

The relation between the numbers l_0, l_1 and the number l (the weight of representation in the basis (6)) is given by a following formula:

$$(l_0,l_1) = (l, l+1),$$

whence it immediately follows that

$$l = \frac{l_0+l_1-1}{2}. \tag{45}$$

As is known [14], if an irreducible representation of the proper Lorentz group $SO_0(1,3)$ is defined by the pair (l_0, l_1), then a conjugated representation is also irreducible and is defined by a pair $\pm(l_0, -l_1)$. Therefore,

$$(l_0, l_1) = (-\dot{l}, \dot{l} + 1).$$

Hence it follows that

$$\dot{l} = \frac{l_0 - l_1 + 1}{2}. \tag{46}$$

For the unitary representations, that is, in the case of principal series representations of the group $SO_0(1,3)$, there exists an analogue of formula (43),

$$V_a f(z) = (a_{12}z + a_{22})^{\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1}\overline{(a_{12}z + a_{22})}^{-\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1} f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right), \tag{47}$$

where $f(z)$ is a measurable function of the Hilbert space $L_2(Z)$, satisfying the condition $\int |f(z)|^2 \,\mathrm{d}z < \infty$, $z = x + \mathrm{i}y$. A totality of all representations $a \to T^\alpha$, corresponding to all possible pairs λ, ρ, is called a principal series of representations of the group $SO_0(1,3)$ and denoted as $\mathfrak{S}_{\lambda,\rho}$. At this point, a comparison of (47) with formula (43) for the spinor representation $\mathfrak{S}_l$ shows that the both formulae have the same structure; only the exponents at the factors $(a_{12}z + a_{22})$, $\overline{(a_{12}z + a_{22})}$ and the functions $f(z)$ are different. In the case of spinor representations the functions $f(z)$ are polynomials $p(z, \bar{z})$ in the spaces $\mathrm{Sym}_{(k,r)}$, and in the case of a representation $\mathfrak{S}_{\lambda,\rho}$ of the principal series $f(z)$ are functions from the Hilbert space $L_2(Z)$.

We know that a representation S_l of the group $SU(2)$ is realized in terms of the functions $t^l_{mn}(u) = \mathrm{e}^{-\mathrm{i}m\varphi} P^l_{mn}(\cos\theta)\, \mathrm{e}^{-\mathrm{i}n\psi}$. We use below the following:

Theorem 1 (Naimark [15]). *The representation S_k of $SU(2)$ is contained in $\mathfrak{S}_{\lambda,\rho}$ not more than one time. At this point, S_k is contained in $\mathfrak{S}_{\lambda,\rho}$, when $\frac{\lambda}{2}$ is one from the numbers $-k, -k+1, \ldots, k$.*

Let us find a relation between the parameters l_0, l_1 in formulae (44) and the parameters λ, ρ of the representation $\mathfrak{S}_{\lambda,\rho}$. The number l_0 is the lowest from the weights k of representations S_k contained in $\mathfrak{S}_{\lambda,\rho}$. Hence it follows that $k \geqslant \left|\frac{\lambda}{2}\right|$ and $l_0 = \left|\frac{\lambda}{2}\right|$. Let us consider the operators

$$\Delta = F_+F_- + F_-F_+ + 2F_3^2 - \left(H_+H_- + H_-H_+ + 2H_3^2\right) = \mathsf{A}^2 - \mathsf{B}^2, \tag{48}$$

$$\Delta' = H_+F_- + H_-F_+ + F_+H_- + F_-H_+ + 4H_3F_3 = \mathsf{AB}. \tag{49}$$

Applying formulae (44), we obtain

$$\Delta\xi_{k\nu} = -2\left(l_0^2 + l_1^2 - 1\right)\xi_{k\nu}, \tag{50}$$

$$\Delta'\xi_{k\nu} = -4\mathrm{i}l_0 l_1 \xi_{k\nu}. \tag{51}$$

On the other hand, calculating infinitesimal operators of the representation $\mathfrak{S}_{\lambda,\rho}$ in the space $L_2(Z)$, we have

$$\mathsf{A}_1 f = \frac{\mathrm{i}}{2}(1 - z^2)\frac{\partial f}{\partial z} - \frac{\mathrm{i}}{2}(1 - \bar{z}^2)\frac{\partial f}{\partial \bar{z}} + \frac{\mathrm{i}}{2}\left[\left(\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1\right)z - \left(-\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1\right)\bar{z}\right] f, \tag{52}$$

$$\mathsf{A}_2 f = \frac{1}{2}(1 + z^2)\frac{\partial f}{\partial z} + \frac{1}{2}(1 + \bar{z}^2)\frac{\partial f}{\partial \bar{z}} - \frac{1}{2}\left[\left(\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1\right)z + \left(-\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1\right)\bar{z}\right] f, \tag{53}$$

$$\mathsf{A}_3 f = \mathrm{i}z\frac{\partial f}{\partial z} - \mathrm{i}\bar{z}\frac{\partial f}{\partial \bar{z}} - \frac{\mathrm{i}}{2}\lambda f, \tag{54}$$

$$\mathsf{B}_1 f = \frac{1}{2}(1-z^2)\frac{\partial f}{\partial z} + \frac{1}{2}(1-\bar{z}^2)\frac{\partial f}{\partial \bar{z}} + \frac{1}{2}\left[\left(\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1\right)z + \left(-\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1\right)\bar{z}\right]f, \tag{55}$$

$$\mathsf{B}_2 f = -\frac{\mathrm{i}}{2}(1+z^2)\frac{\partial f}{\partial z} + \frac{\mathrm{i}}{2}(1+\bar{z}^2)\frac{\partial f}{\partial \bar{z}} + \frac{\mathrm{i}}{2}\left[\left(\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1\right)z - \left(-\frac{\lambda}{2}+\mathrm{i}\frac{\rho}{2}-1\right)\bar{z}\right]f, \tag{56}$$

$$\mathsf{B}_3 f = z\frac{\partial f}{\partial z} + \bar{z}\frac{\partial f}{\partial \bar{z}} + \left(1-\mathrm{i}\frac{\rho}{2}\right)f. \tag{57}$$

Substituting the latter expressions into (48) and (49), we find that

$$\Delta f(z) = -2\left[\left(\frac{\lambda}{2}\right)^2 - \left(\frac{\rho}{2}\right)^2 - 1\right]f(z), \qquad \Delta' f(z) = -\lambda\rho f(z).$$

The comparison of these formulae with (50) and (51) shows that

$$\left(\frac{\lambda}{2}\right)^2 - \left(\frac{\rho}{2}\right)^2 = l_0^2 + l_1^2, \tag{58}$$

$$\lambda\rho = 4\mathrm{i}l_0 l_1. \tag{59}$$

Let $l_0 \neq 0$; since $l_0 = \left|\frac{\lambda}{2}\right|$, then from (59) it follows that

$$l_1 = -\mathrm{i}(\operatorname{sign}\lambda)\frac{\rho}{2}.$$

If $l_0 = 0$ and, therefore, $\lambda = 0$, from (58) we obtain

$$l_1 = \pm\mathrm{i}\frac{\rho}{2}.$$

Thus, the numbers l_0, l_1 λ, ρ are related by the formulae

$$l_0 = \left|\frac{\lambda}{2}\right|, \qquad l_1 = -\mathrm{i}(\operatorname{sign}\lambda)\frac{\rho}{2} \quad \text{if } \lambda \neq 0, \qquad l_0 = 0, \qquad l_1 = \pm\mathrm{i}\frac{\rho}{2} \quad \text{if } \lambda = 0.$$

On the other hand, we see from (7) that Laplace–Beltrami operators $\mathsf{X}^2 = -l(l+1)$ and $\mathsf{Y}^2 = -\dot{l}(\dot{l}+1)$ contain Casimir operators $\Delta = \mathsf{A}^2 - \mathsf{B}^2$ and $\Delta' = \mathsf{A}\cdot\mathsf{B}$ as real and imaginary parts:

$$\mathsf{X}^2 = \tfrac{1}{4}\Delta + \tfrac{1}{2}\mathrm{i}\Delta' = -l(l+1), \qquad \mathsf{Y}^2 = \tfrac{1}{4}\Delta - \tfrac{1}{2}\mathrm{i}\Delta' = -\dot{l}(\dot{l}+1).$$

Taking into account in the latter operators formulae (50) and (51), we arrive at relations (45) and (46). Relations (45) and (46) define a relation between parameters l_0, l_1 of the Gel'fand–Naimark basis (44) and parameters $l, \dot{l}$ of the Van der Waerden basis (6).

As is known, all the unitary representations of the group $SO_0(1,3)$ are infinite dimensional. The group $SO_0(1,3)$ is non-compact and one from its real forms; the group $SU(1,1)$ is also non-compact group involving unitary infinite-dimensional representations. In the previous section, it has been shown that the matrix elements of $SO_0(1,3)$ are defined via the addition theorem for the matrix elements of the subgroups $SU(2)$ and $SU(1,1)$. This factorization allows us to separate explicitly in the matrix element all the parameters changing in infinite limits.

In such a way, using theorem 1, formulae (30), (37) and (45), we find that matrix elements of the principal series representations of the group $SO_0(1,3)$ have the form

$$\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\mathfrak{g}) = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}Z_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0} = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}$$
$$\times \sum_{t=-l_0}^{l_0} \mathrm{i}^{m-t}\sqrt{\Gamma(l_0-m+1)\Gamma(l_0+m+1)\Gamma(l_0-t+1)\Gamma(l_0+t+1)}$$

$$\times \cos^{2l_0}\frac{\theta}{2}\tan^{m-t}\frac{\theta}{2}$$
$$\times \sum_{j=\max(0,t-m)}^{\min(l_0-m,l_0+t)} \frac{\mathrm{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l_0-m-j+1)\Gamma(l_0+t-j+1)\Gamma(m-t+j+1)}$$
$$\times \sqrt{\Gamma\left(\tfrac{1}{2}+\mathrm{i}\rho-n\right)\Gamma\left(\tfrac{1}{2}+\mathrm{i}\rho+n\right)\Gamma\left(\tfrac{1}{2}+\mathrm{i}\rho-t\right)\Gamma\left(\tfrac{1}{2}+\mathrm{i}\rho+t\right)}\cosh^{-1+2\mathrm{i}\rho}\frac{\tau}{2}\tanh^{n-t}\frac{\tau}{2}$$
$$\times \sum_{s=\max(0,t-n)}^{\infty} \frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma\left(\frac{1}{2}+\mathrm{i}\rho-n-s\right)\Gamma\left(\frac{1}{2}+\mathrm{i}\rho+t-s\right)\Gamma(n-t+s+1)}, \tag{60}$$

where $l_0 = \left|\frac{\lambda}{2}\right|$ and $\frac{\lambda}{2}$ is one from the numbers $-k, -k+1, \dots, k$. It is obvious that $\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\mathfrak{g})$ cannot be attributed as matrix elements to single irreducible representation. From the latter expression it follows that relativistic spherical functions $f(\mathfrak{g})$ of the principal series can be defined by means of the function

$$\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\mathfrak{g}) = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)}Z_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\cos\theta^{\mathrm{c}})\,\mathrm{e}^{-n(\varepsilon+\mathrm{i}\psi)}, \tag{61}$$

where

$$Z_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\cos\theta^{\mathrm{c}}) = \sum_{t=-l_0}^{l_0} P_{mt}^{l_0}(\cos\theta)\mathfrak{P}_{tn}^{-\frac{1}{2}+\mathrm{i}\rho}(\cosh\tau).$$

In the case of relativistic spherical functions $f(\varphi^{\mathrm{c}},\theta^{\mathrm{c}})$ we have

$$Z^{m}_{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\cos\theta^{\mathrm{c}}) = \sum_{t=-l_0}^{l_0} P_{mt}^{l_0}(\cos\theta)\mathfrak{P}^{t}_{-\frac{1}{2}+\mathrm{i}\rho}(\cosh\tau), \tag{62}$$

where $\mathfrak{P}^{t}_{-\frac{1}{2}+\mathrm{i}\rho}(\cosh\tau)$ are *conical functions* (see [37]). In this case our result agrees with the paper [38], where matrix elements (eigenfunctions of Casimir operators) of non-compact rotation groups are expressed in terms of conical and spherical functions (see also [21]).

When ρ is a cleanly imaginary number, $\rho = \mathrm{i}\sigma$, we have

$$T^{\alpha}f(z) = |a_{12}z+a_{22}|^{-2-\sigma}f\left(\frac{a_{11}z+a_{21}}{a_{12}z+a_{22}}\right).$$

This formula defines a unitary representation $a \to T^{\alpha}$ of supplementary series $\mathfrak{D}_\sigma$ of the group $SO_0(1,3)$. In its turn, for the supplementary series $\mathfrak{D}_\sigma$ the following theorem holds.

Theorem 2 (Naimark [15]). *The representation S_k of $SU(2)$ is contained in $\mathfrak{S}_{\lambda,\rho}$ when k is an integer number. In this case, S_k is contained in $\mathfrak{D}_\sigma$ exactly one time.*

We see that $\frac{\lambda}{2} = 0$ should be one from the numbers $-k, -k+1, \dots, k$, therefore, when k is integer.

Let us find a relation between the parameters l_0, l_1 in (44) and the parameter σ of $\mathfrak{D}_\sigma$. First, the lowest weight l_0 from the weights k of the representations S_k, contained in $\mathfrak{D}_\sigma$, is equal to zero, that is, $l_0 = 0$. With the aim to define the parameter l_1 let us consider again the Casimir operator $\Delta = \mathsf{A}^2 - \mathsf{B}^2$. Calculating this operator with the help of formulae (44) and (52)–(57), where $\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1$ and $-\frac{\lambda}{2} + \mathrm{i}\frac{\rho}{2} - 1$ should be replaced by $-\frac{\sigma}{2} - 1$ and $-\frac{\sigma}{2} - 1$, we obtain

$$\Delta\xi_{k\nu} = -2\left(l_1^2-1\right)\xi_{k\nu}, \qquad \Delta f(z) = -2\left[\left(\frac{\sigma}{2}\right)^2 - 1\right]f(z).$$

Hence it follows that $l_1^2=\left(\frac{\sigma}{2}\right)^2$ and $l_1=\pm\frac{\sigma}{2}$ (the choice of the sign is not important). Thus, for the supplementary series the relations

$$l_0=0,\qquad l_1=\pm\frac{\sigma}{2}$$

hold. In the case of $\mathfrak{D}_\sigma$, Laplace–Beltrami operators X^2 and Y^2 coincide with each other, $\mathsf{X}^2=\mathsf{Y}^2=\mathsf{A}^2-\mathsf{B}^2$. This means that we come here to representations of $SO_0(1,3)$ restricted to the subgroup $SU(1,1)$.

Thus, matrix elements of supplementary series appear as a particular case of the matrix elements of the principal series at $l_0=0$ and $\rho=\mathrm{i}\sigma$:

$$\begin{aligned}\mathfrak{M}_{mn}^{-\frac{1}{2}-\sigma}(\mathfrak{g})&=\mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}Z_{mn}^{-\frac{1}{2}-\sigma}=\mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}\\&\times\sqrt{\Gamma\left(\tfrac{1}{2}-\sigma-n\right)\Gamma\left(\tfrac{1}{2}-\sigma+n\right)\Gamma\left(\tfrac{1}{2}-\sigma-m\right)\Gamma\left(\tfrac{1}{2}-\sigma+m\right)}\cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\\&\times\sum_{s=\max(0,m-n)}^{\infty}\frac{\tanh^{2s}\frac{\tau}{2}}{\Gamma(s+1)\Gamma\left(\frac{1}{2}-\sigma-n-s\right)\Gamma\left(\frac{1}{2}-\sigma+m-s\right)\Gamma(n-m+s+1)}.\end{aligned}\tag{63}$$

Or

$$\mathfrak{M}_{mn}^{-\frac{1}{2}-\sigma}(\mathfrak{g})=\mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)}\mathfrak{P}_{mn}^{-\frac{1}{2}-\sigma}(\cosh\tau)\,\mathrm{e}^{-n(\varepsilon+\mathrm{i}\psi)},$$

that is, the hyperspherical function $Z_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\cos\theta^c)$ in the case of supplementary series is reduced to the Jacobi function $\mathfrak{P}_{mn}^{-\frac{1}{2}-\sigma}(\cosh\tau)$. For the relativistic spherical functions $f(\varphi^c,\theta^c)\sim f(\varphi^c,\tau)$ of supplementary series we obtain

$$\mathfrak{M}^m_{-\frac{1}{2}-\sigma}(\mathfrak{g})=\mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)}\mathfrak{P}^m_{-\frac{1}{2}-\sigma}(\cosh\tau).$$

Let us now express the relativistic spherical function $\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho}(\mathfrak{g})$ of the principal series representations of $SO_0(1,3)$ via the hypergeometric function. Using formulae (60), (41), (42) and (31), (32), we find

$$\begin{aligned}\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\mathfrak{g})&=\mathrm{e}{-}m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)\sqrt{\frac{\Gamma(l_0+m+1)\Gamma\left(\mathrm{i}\rho-n+\frac{1}{2}\right)}{\Gamma(l_0-m+1)\Gamma\left(\mathrm{i}\rho+n+\frac{1}{2}\right)}}\\&\times\cos^{2l_0}\frac{\theta}{2}\cosh^{-1+2\mathrm{i}\rho}\frac{\tau}{2}\sum_{t=-l_0}^{l_0}\mathrm{i}^{m-t}\tan^{m-t}\frac{\theta}{2}\tanh^{t-n}\frac{\tau}{2}\\&\times{}_2F_1\left(\left.\begin{matrix}m-l_0,-t-l_0\\m-t+1\end{matrix}\right|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\left.\begin{matrix}t-\mathrm{i}\rho+\frac{1}{2},-n-\mathrm{i}\rho+\frac{1}{2}\\t-n+1\end{matrix}\right|\tanh^2\frac{\tau}{2}\right),\qquad m\geqslant n;\end{aligned}$$

$$\begin{aligned}\mathfrak{M}_{mn}^{-\frac{1}{2}+\mathrm{i}\rho,l_0}(\mathfrak{g})&=\mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}\sqrt{\frac{\Gamma(l_0-m+1)\Gamma\left(\mathrm{i}\rho+n+\frac{1}{2}\right)}{\Gamma(l_0+m+1)\Gamma\left(\mathrm{i}\rho-n+\frac{1}{2}\right)}}\\&\times\cos^{2l_0}\frac{\theta}{2}\cosh^{-1+2\mathrm{i}\rho}\frac{\tau}{2}\sum_{t=-l_0}^{l_0}\mathrm{i}^{t-m}\tan^{t-m}\frac{\theta}{2}\tanh^{n-t}\frac{\tau}{2}\\&\times{}_2F_1\left(\left.\begin{matrix}t-l_0,-m-l_0\\t-m+1\end{matrix}\right|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\left.\begin{matrix}n-\mathrm{i}\rho+\frac{1}{2},-t-\mathrm{i}\rho+\frac{1}{2}\\n-t+1\end{matrix}\right|\tanh^2\frac{\tau}{2}\right),\qquad n\geqslant m.\end{aligned}$$

For the supplementary series we have

$$\mathfrak{M}_{mn}^{-\frac{1}{2}-\sigma}(\mathfrak{g}) = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}\sqrt{\frac{\Gamma\big(m-\sigma+\frac{1}{2}\big)\Gamma\big(-n-\sigma+\frac{1}{2}\big)}{\Gamma\big(-m-\sigma+\frac{1}{2}\big)\Gamma\big(n-\sigma+\frac{1}{2}\big)}}$$
$$\times\cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{m-n}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix}m+\sigma+\frac{1}{2},\,-n+\sigma+\frac{1}{2}\\ m-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\qquad m\geqslant n;$$

$$\mathfrak{M}_{mn}^{-\frac{1}{2}-\sigma}(\mathfrak{g}) = \mathrm{e}^{-m(\epsilon+\mathrm{i}\varphi)-n(\varepsilon+\mathrm{i}\psi)}\sqrt{\frac{\Gamma\big(n-\sigma+\frac{1}{2}\big)\Gamma\big(-m-\sigma+\frac{1}{2}\big)}{\Gamma\big(-n-\sigma+\frac{1}{2}\big)\Gamma\big(m-\sigma+\frac{1}{2}\big)}}$$
$$\times\cosh^{-1-2\sigma}\frac{\tau}{2}\tanh^{n-m}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix}n+\sigma+\frac{1}{2},\,-m+\sigma+\frac{1}{2}\\ n-m+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\qquad n\geqslant m.$$

In like manner we can define conjugated spherical functions $f(\mathfrak{g}) = \mathfrak{M}_{mn}^{-\frac{1}{2}-\mathrm{i}\rho,l_0}(\mathfrak{g})$ and $f(\dot{\varphi}^c,\dot{\theta}^c) = \mathfrak{M}^m_{-\frac{1}{2}-\mathrm{i}\rho,l_0}(\dot{\varphi}^c,\dot{\theta}^c,0)$ of the principal series $\mathfrak{S}_{\lambda,\rho}$, since a conjugated representation of $SO_0(1,3)$ is defined by the pair $\pm(l_0,-l_1)$. It is obvious that in the case of supplementary series $\mathfrak{D}_\sigma$ we arrive at the same functions $\mathfrak{M}_{mn}^{-\frac{1}{2}-\sigma}(\mathfrak{g})$.

References

[1] Dolginov A Z 1956 Relativistic spherical functions *Zh. Eksp. Teor. Fiz.* **30** 746–55

[2] Dolginov A Z and Toptygin I N 1959 Relativistic spherical functions. II *Zh. Eksp. Teor. Fiz.* **37** 1441–51

[3] Dolginov A Z and Moskalev A N 1959 Relativistic spherical functions. III *Zh. Eksp. Teor. Fiz.* **37** 1697–707

[4] Eskin L D 1959 On the theory of relativistic spherical functions *Nauchn. Dokl. Vys. Shkoly* **2** 95–7

[5] Fock V A 1935 Zur theorie des wasserstoffatoms *Zs. F. Phys.* **98** 145–54

[6] Eskin L D 1961 On the matrix elements of irreducible representations of the Lorentz group *Izv. Vuzov, Matem.* **6** 179–84

[7] Golodets V Ya 1961 Matrix elements of irreducible unitary and spinor representations of the homogeneous Lorentz group *Vesti Akad. Navuk Belarusskoi SSR* **1** 19–28

[8] Ström S 1965 On the matrix elements of a unitary representation of the homogeneous Lorentz group *Ark. Fysk.* **29** 467–83

[9] Ström S 1967 A note on the matrix elements of a unitary representation of the homogeneous Lorentz group *Ark. Fysk.* **33** 465–9

[10] Ström S 1968 Matrix elements of the supplementary series of unitary representations of $SL(2,C)$ *Ark. Fysk.* **38** 373–81

[11] Sciarrino A and Toller M 1967 Decomposition of the unitary irreducible representations of the group $SL(2,C)$ restricted to the subgroup $SU(1,1)$ *J. Math. Phys.* **8** 1252–65

[12] Verdiev I A and Dadashev L A 1967 Matrix elements of the Lorentz group unitary representation *Yad. Fiz.* **6** 1094–9

[13] Kolomytsev V I 1970 The reduction of the irreducible unitary representations of the group $SL(2,C)$ restricted to the subgroup $SU(1,1)$. The additional series *Teor. Mat. Fiz.* **2** 210–29

[14] Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Groups and their Applications* (Oxford: Pergamon)

[15] Naimark M A 1964 *Linear Representations of the Lorentz Group* (London: Pergamon)

[16] Smorodinsky Ya A and Huszar M 1970 Representations of the Lorentz group and the generalization of helicity states *Teor. Mat. Fiz.* **4** 328–40

[17] Huszar M 1985 Spherical functions of the Lorentz group on the hyperboloids *Acta Phys. Hung.* **58** 175–85

[18] Huszar M 1988 Addition theorems for the spherical functions of the Lorentz group *Acta Phys. Hung.* **64** 361–78

[19] Andrews M and Gunson J 1964 Complex angular momenta and many-particle states. I. Local representations of the rotation group *J. Math. Phys.* **5** 1391–400

[20] Gel'fand I M and Shapiro Z Ya 1952 Representations of the rotation group of three-dimensional space and their applications *Usp. Mat. Nauk* **7** 3–117

[21] Vilenkin N Ya 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society)

[22] Bargmann V 1947 Irreducible unitary representations of the Lorentz group *Ann. Math.* **48** 568–640

[23] Kihlberg A 1970 Fields on a homogeneous space the Poincaré group *Ann. Inst. Henri Poincaré* **13** 57–76
[24] Toller M 1996 Free quantum fields on the Poincaré group *J. Math. Phys.* **37** 2694–730
[25] Drechsler W 1997 Geometro-stochastically quantized fields with internal spin variables *J. Math. Phys.* **38** 5531–58
[26] Gitman D M and Shelepin A L 2001 Fields on the Poincaré group: arbitrary spin description and relativistic wave equations *Int. J. Theor. Phys.* **40** 603–84
[27] Varlamov V V 2004 Relativistic wavefunctions on the Poincaré group *J. Phys. A: Math. Gen.* **37** 5467–76
[28] Varlamov V V 2005 Maxwell field on the Poincaré group *Int. J. Mod. Phys.* A **20** 4095–112
[29] Agamaliev A K, Atakishiev N M and Verdiev I A 1969 Invariant expansion of solutions of relativistic equations *Yad. Fiz.* **9** 201–11
[30] Varlamov V V 2003 General solutions of relativistic wave equations *Int. J. Theor. Phys.* **42** 583–633
[31] Biedenharn L C, Braden H W, Truini P and van Dam H 1988 Relativistic wavefunctions on spinor spaces *J. Phys. A: Math. Gen.* **21** 3593–610
[32] van der Waerden B L 1932 *Die Gruppentheoretische Methode in der Quantenmechanik* (Berlin: Springer)
[33] Vilenkin N Ya and Klimyk A U 1991–1993 *Representations of Lie Groups and Special Functions* vol 1–3 (Dordrecht: Kluwer)
[34] Hollman W J and Biedenharn L C 1966 Complex angular momenta and the groups $SU(1,1)$ and $SU(2)$ *Ann. Phys., NY* **39** 1–42
[35] Chandra Harish 1947 Infinite irreducible representations of the Lorentz group *Proc. R. Soc.* A **189** 372–401
[36] Gel'fand I M and Yaglom A M 1948 General relativistic-invariant equations and infinite-dimensional representations of the Lorentz group *Zh. Eksp. Teor. Fiz.* **18** 703–33
[37] Bateman H and Erdélyi A 1953 *Higher Transcendental Functions* vol I (New York: McGraw-Hill)
[38] Fischer J, Niederle J and Raczka R 1966 Generalized spherical functions for the noncompact rotation groups *J. Math. Phys.* **7** 816–21